D-modules and b-functions: a survey.

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Our purpose in this course is to give a survey of the various aspects, algebraic, analytic and formal, of the functional equations which are satisfied by the powers f^s of a function f. There is a non zero polynomial in one variable $b_f(s)$ called the Bernstein-Sato polynomial of f or b-function for short, involved in an equation of the form :

$$P(s)f^{s+1} = b(s)f^s$$

where $P(s)$ is a linear partial differential operator depending on a parameter s.

Since this course is intended to be useful for newcomers to the subject we shall rely on a number of basic examples and beforehand give a short summary of basic facts about D-modules. These are systems of linear partial differential equation treated from an algebraic view point as coherent modules over the sheaf of linear partial differential operators.

Before giving an idea of the proof of the existence theorem, we give an account of motivations, applications and most strikening results following from this existence. The problem, of meromorphic continuation of the distribution f_+^s is the first historical motivation as mentionned in [2]. During the 70's it was proved that the zeroes of b_f are rational negative (cf [6], and [4]), and a narrow link was discovered, between the roots of the b-function and the eigenvalues of the monodromy of f, by Malgrange in the isolated singularity case [6], and Malgrange Kashiwara in general. See[4]. A fundamental fact, following from the functional equation, is that the module $\mathcal{O}[\frac{1}{f}]$ is holonomic. This has a vast generalisation to all modules of local cohomology.

The second part of this course will be devoted to examples, and computational aspects: case of quasi or of semiquasi homogeneous polynomials with isolated singularities, of hyperplane arrangements. We shall also describe another example of high historical importance, due to Mikio Sato, the semi-invariants of prehomogeneous actions on a complex vector space, of an algebraic group, and especially of a reductive group. see [5].

We refer principally to our course [1] and a forthcoming amended version. See also for $\mathcal{D}$-module theory [3], and also the complete bibliography in these references.

References

[1] Granger, Michel, *Bernstein-Sato polynomials and functional equations*, in Algebraic approach to differential equations,World Sci. Publ., Hackensack, NJ 2010. pp 225–291.

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[3] M. Granger and Ph. Maisonobe, *A basic course on differential modules.* In: Eléments de la théorie des systèmes différentiels, tome 1: $\mathcal{D}_X$-modules cohérents. Travaux en cours n° 46 Hermann, 1993, tome I, pp. 103-167.

[4] M. Kashiwara, *B-functions and holonomic systems.* Inventiones Mathematicae **38**, 33-53 (1976).

[5] T. Kimura, *Introduction to Prehomogeneous Vector Spaces.* Translations of Mathematical monographs, vol 215, Amer. Math. Soc. (2003)

[6] B. Malgrange, *Le polynôme de Bernstein d'une singularité isolée.* Lectures notes in Mathematics **459**, 98-119 Springer (1974).